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FIBRATIONS OF SPHERES BY GREAT SPHERES OVER DIVISION ALGEBRAS AND THEIR DIFFERENTIABILITY

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0. Introduction

Fibrations of $\mathbb{S}^{2n-1}$ by great $(n-1)$-spheres arise in the theory of Blaschke manifolds; see Gluck-Warner-Yang [4], in particular §2, p. 1043. Their Theorem B, p. 1041, states that every *differentiable* fibration of this kind is *topologically* equivalent to the fibration of $\mathbb{S}^{2n-1}$ determined by a division algebra. (This division algebra is obtained by a certain linearization process; see Yang [15], Gluck-Warner-Yang [4, §6, p. 1056] and [9, §3, 3.2]. Let us call it the "infinitesimal division algebra". It should be noted that in general it is neither associative nor alternative.)

Here we answer the natural question: When is the fibration of $\mathbb{S}^{2n-1}$ by great $(n-1)$-spheres determined by a division algebra *differentiable* (as a locally trivial fiber bundle)? This turns out to be the case only for the classical Hopf fibrations, which are determined by the classical division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ (see Theorem 1.3 below). This result contradicts Theorem 2 of Yang [15]; his proof contains a fallacy (see [9, 2.10]).

It is possible to construct examples of differentiable fibrations of $\mathbb{S}^{2n-1}$ by great $(n-1)$-spheres for which the infinitesimal division algebras are not classical; this shows that the approach of Gluck-Warner-Yang [4] is really only topological (as they point out in Remark 1, p. 1075, without further explanation), and it invalidates Theorem 4 of Yang [15].

However, we still conjecture that every differentiable fibration of $\mathbb{S}^{2n-1}$ by great $(n-1)$-spheres is *differentiably* equivalent to the classical Hopf fibration of the same dimension. For $n = 1, 2$ this is more or less trivially true; for $n = 4$, it has been proved in [9]. For the remaining case $n = 8$, the problem seems to be open.

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The topic of this paper is connected with the theory of topological projective planes; see §2 below and [9, §§1, 2].

1. Fibrations determined by division algebras

1.1. Division algebras. A (real) division algebra D of finite dimension n is a real vector space $D = \mathbb{R}^n$ equipped with a bilinear multiplication $(x,y) \mapsto x \cdot y \colon D^2 \to D$ which satisfies

(i) every left multiplication map

$$\lambda_a \colon D \to D \colon x \mapsto a \cdot x$$

with $0 \neq a \in D$ is invertible, i.e., $\lambda_a \in \mathrm{GL}_n \mathbb{R}$.

(ii) there is a "unit element" $1 \in D$ with $1 \cdot x = x = x \cdot 1$ for every $x \in D$.

Note that the multiplication is not required to be associative or alternative.

As a consequence of (i), every nonzero right multiplication map

$$\rho_a \colon D \to D \colon x \mapsto x \cdot a$$

is invertible as well. We denote the inverse operations by

$$a\backslash b = \lambda_a^{-1}(b) \quad \text{and} \quad b/a = \rho_a^{-1}(b)$$

for $a, b \in D$ with $a \neq 0$; in other words, $a\backslash b$ (resp. b/a) is the unique solution x of the equation $a \cdot x = b$ (resp. $x \cdot a = b$).

The classical examples are, of course, $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (the quaternions) and $\mathbb{O}$ (the octonions). But besides these there is a plethora of other real division algebras. For just a few families of examples, cf. Yang [15], [6], [8, 2.6, §3], [7, §4, p. 214]; the latter examples are also found in Benkart-Osborn [1]. See also the references in [5, 7.2].

1.2. Fibrations determined by division algebras. Let D be a real division algebra of dimension n. Define n-dimensional subspaces of $D \oplus D = \mathbb{R}^{2n}$ as follows:

$$U_a = \{(x, a \cdot x) \mid x \in D\} \quad \text{for } a \in D, \qquad U_\infty = \{0\} \times D.$$

Then the intersections $U_a \cap \mathbb{S}^{2n-1}$ for $a \in D \cup \{\infty\}$ are the fibers of a fibration π of the unit sphere $\mathbb{S}^{2n-1}$ of $\mathbb{R}^{2n}$ into great $(n-1)$-spheres (we deviate slightly from Yang [15, Theorem 2, p. 580] by interchanging the first and second coordinates). The classical division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ lead to the Hopf fibrations.

The fibration π obtained in this way from any division algebra D is always a topological locally trivial fiber bundle (see the proof of Proposition 2.5 in [9]). Here we are concerned with the question: When is π a *differentiable* fiber bundle? Theorem 2 of Yang [15] asserts that this is always the case. This assertion is drastically refuted by Theorem 1.3 below (for the fallacy in Yang's proof see [9, 2.10]), which means that from the multitude of finite-dimensional real division algebras, a differentiable fiber bundle is obtained only in the classical cases:

1.3. Theorem. *The fibration π determined by a real division algebra D of finite dimension is a differentiable locally trivial fiber bundle if and only if D is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.*

Proof. The Hopf fibrations are known to be differentiable locally trivial fiber bundles. Conversely, assume π to be differentiable. We use the differentiability criterion given in [9, 2.5] for arbitrary fibrations of $\mathbb{S}^{2n-1}$ by great $(n-1)$-spheres (not necessarily determined by division algebras). It states that the map

$$\bar{\gamma}\colon D \times (D\backslash\{0\}) \to \mathrm{End}_{\mathbb{R}}(D)\colon (x,y) \mapsto \begin{cases} \lambda_{y/x}^{-1} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

must be differentiable (even at $x = 0$). In particular, for every fixed vector $v \neq 0$ the map

$$\bar{\gamma}_v\colon D \to \mathrm{End}_{\mathbb{R}}(D)\colon x \mapsto \begin{cases} \lambda_{v/x}^{-1} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

is differentiable, with differential $d_0\bar{\gamma}_v$ at $x = 0$. For $t \in \mathbb{R}\backslash\{0\}$ and $v, x \in D$ with $x \neq 0$, bilinearity of the multiplication implies $v/(tx) = t^{-1}(v/x)$, hence $\lambda_{v/(tx)} = t^{-1}\lambda_{v/x}$ and $\lambda_{v/(tx)}^{-1} = t\lambda_{v/x}^{-1}$. This yields

$$\begin{aligned} d_0\bar{\gamma}_v(x) &= \frac{d}{dt}\bar{\gamma}_v(tx)\Big|_{t=0} = \frac{d}{dt}\lambda_{v/(tx)}^{-1}\Big|_{t=0} \\ &= \frac{d}{dt}(t\lambda_{v/x}^{-1})\Big|_{t=0} = \lambda_{v/x}^{-1} = \bar{\gamma}_v(x). \end{aligned}$$

Thus $\bar{\gamma}_v$ is linear, by the linearity of a differential, and

$$x \mapsto \bar{\gamma}_v(x)(z) = \begin{cases} (v/x)\backslash z & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

is a linear endomorphism of $D = \mathbb{R}^n$ for every $z \in D$. In other words we have obtained the identity

$$(v/(x + x'))\backslash z = (v/x)\backslash z + (v/x')\backslash z,$$

which holds for $v, x, x', z \in D$ with $v, x, x', x + x'$ all distinct from zero. Now the proof is completed by the following lemma, which requires only the special case $x' = 1$, $v = x+1 = (1+1/x)\cdot x$ (and hence $v/x = 1+1/x$) of the identity above.

1.4. Lemma. *Let D be a real finite-dimensional division algebra which satisfies the identity*

$$z = (1 + 1/x)\backslash z + (x + 1)\backslash z$$

for $x, z \in D$, $x \neq 0, -1$. Then D is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

Proof. Replacing z by $(x + 1) \cdot z$ gives

$$x \cdot z + z = (x + 1) \cdot z = (1 + 1/x)\backslash(x \cdot z + z) + z,$$

hence $x \cdot z = (1+1/x)\backslash(x \cdot z + z)$, which is equivalent to $(1+1/x)\cdot(x \cdot z) = x \cdot z + z$. This yields $(1/x)\cdot(x \cdot z) = z$, i.e., D has the left inverse property (cf. Hughes-Piper [10, p. 135] or Pickert [12, p. 106]; note that the special case $x \cdot z = 1$ shows $1/x = x\backslash 1$). By a result of Skornyakov-San Soucie (see Hughes-Piper [10, Theorem 6.16, p. 140] or Pickert [12, 6.16, p. 182]), D is an alternative division algebra, hence isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ by well-known theorems of Frobenius (cf. Palais [11] or Ebbinghaus et al. [3, p. 161]) and Zorn [16] (cf. also Ebbinghaus et al. [3, p. 178] or Pickert [12, p. 177]).

2. Differentiable projective planes over division algebras

2.1. A *differentiable projective plane* is a projective plane whose point set P and line set $\mathscr{L}$ are endowed with the structure of a differentiable manifold of positive dimension such that the points on a fixed line and dually the pencil of lines through a fixed point form submanifolds and such that the operations $\vee$ and $\wedge$ of joining distinct points and intersecting distinct lines are differentiable; cf. Breitsprecher [2]. We shall consider lines as subsets of the point set (by identification with the set of incident points).

It is a conjecture of Betten that the four classical planes over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the only differentiable projective planes; here we establish a special case of this conjecture.

2.2. Theorem. *The only differentiable projective planes which are translation planes as well as dual translation planes are the classical projective planes over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.*

2.3. Explanations. A projective plane is called a *translation plane* if there is a line L such that the group of all translations with axis L acts transitively on the points not on L; the line L is then called a "translation line". The dual condition, i.e., the existence of a "translation point", characterizes the dual translation planes. The projective planes which are translation planes as well as dual translation planes are known as the planes of Lenz-type (at least) V (cf. Pickert [12, 3.14, p. 70]); these are precisely the planes which can be coordinatized by (nonassociative) division rings; see below.

Proof of Theorem 2.2. Let L be a translation line. If some point not on L is a translation point, then every point is a translation point; cf. Hughes-Piper [10, Theorem 4.20, p. 101]. Hence we may assume that we have a translation point v on L. We pick points o, u, e such that o, u, v, e form a nondegenerate quadrangle with $L = u \vee v$, and we put

$$w = (o \vee e) \wedge L.$$

Coordinatization of the plane with respect to o, u, v, e amounts to the following: On $D := (o \vee e)\backslash\{w\}$ we define an addition and a multiplication by

$$x + y := ((xu \wedge ov)w \wedge yv)u \wedge oe,$$
$$x \cdot y := ((xu \wedge ev)o \wedge yv)u \wedge oe,$$

for $x, y \in D$; here we have used the abbreviation $xu = x \vee u$ for the line joining x and u. Then $(D, +, \cdot)$ is a (nonassociative) division ring, or, in other terminology, a semifield; see Hughes-Piper [10, Theorem 6.9, p. 134] or Pickert [12, 3.3.8 and 3.3.9, p. 101] or Stevenson [14, 13.2.1, p. 372]. In particular, $(D, +)$ is an abelian group, and for $a \in D\backslash\{0\}$ the left and right multiplication maps $\lambda_a = (x \mapsto a \cdot x)$ and $\rho_a = (x \mapsto x \cdot a)$ are automorphisms of $(D, +)$ (this expresses the distributivity and divisibility properties of the multiplication).

Differentiability of join and intersection implies that the algebraic operations of D and their inverses are differentiable. In particular, $(D, +)$ is an abelian Lie group, and $(D, +) \cong (\mathbb{R}^n, +)$ for some natural number n (cf. also Salzmann [13, 7.23]) since the left multiplications λ_a with $0 \neq a \in D$ form a transitive set of automorphisms. By continuity, the automorphisms λ_a and ρ_a are $\mathbb{R}$-linear, and the multiplication is $\mathbb{R}$-bilinear. Hence D is a real division algebra as defined in 1.1.

The point set A of the affine plane with L as the line at infinity is identified with $D \oplus D = \mathbb{R}^{2n}$ by mapping a point p not on L onto the pair $(pv \wedge oe, pu \wedge oe)$. The lines of the affine plane are then just the subspaces U_a, $a \in D \cup \{\infty\}$, as in 1.2 together with their cosets in $D \oplus D = \mathbb{R}^{2n}$.

From this point on we indicate two ways to prove Theorem 2.2. The first one involves the fibration determined by D. The map

$$\pi\colon A\backslash\{o\} \to L\colon p \mapsto po \wedge L$$

is the projection map of a differentiable fiber bundle whose fibers are the subsets $U_a\backslash\{0\} \cong D\backslash\{0\} \cong \mathbb{R}^n\backslash\{0\}$ for $a \in D \cup \{\infty\}$; local trivializations are given by

$$\begin{aligned} A\backslash U_0 &\to (L\backslash\{u\}) \times (D\backslash\{0\}) \\ p &\mapsto (po \wedge L, pu \wedge oe) \end{aligned}$$

and

$$\begin{aligned} A\backslash U_\infty &\to (L\backslash\{v\}) \times (D\backslash\{0\}) \\ p &\mapsto (po \wedge L, pv \wedge oe). \end{aligned}$$

In our coordinates, with A identified with $D \oplus D$, these trivializations are just the maps $(x,y) \mapsto (\pi(x,y), y)$ and $(x,y) \mapsto (\pi(x,y), x)$.

We now consider the restriction of π to the unit sphere $\mathbb{S}^{2n-1}$ of $A = \mathbb{R}^{2n}$, i.e., the map

$$\pi\colon \mathbb{S}^{2n-1} \to L\colon p \mapsto po \wedge L.$$

The fibers of this restriction are the subsets $U_a \cap \mathbb{S}^{2n-1} \cong \mathbb{S}^{n-1}$; thus we get precisely the fibration of $\mathbb{S}^{2n-1}$ determined by the division algebra D according to 1.2. Local trivializations for this restriction are obtained by appending the radial projection of $D\backslash\{0\} \cong \mathbb{R}^n\backslash\{0\}$ onto $\mathbb{S}^{n-1}$ to the local trivializations above, so we still have a differentiable fiber bundle. Therefore the assertion of Theorem 2.2 follows from Theorem 1.3.

(We remark that the trivializations in Yang [15, Theorem 2] can be obtained as an algebraic transcription of these simple geometric ideas; see [9, 2.9].)

The second (more direct) approach is based on the following geometric calculation using our identification of the affine plane with $D \oplus D$: for $x, y, z \in D$ with $y \neq 0$ we have

$$((x,y) \vee (0,0)) \wedge ((0,z) \vee u) = U_{y/x} \wedge (D \times \{z\}) = (\lambda_{y/x}^{-1}(z), z)$$

if $x \neq 0$, and

$$((0,y) \vee (0,0)) \wedge ((0,z) \vee u) = U_\infty \wedge (D \times \{z\}) = (0,z).$$

Hence differentiability of join and intersection implies that the map $\bar{\gamma}$ in the proof of Theorem 1.3 is differentiable, and we can proceed as in that proof.

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